Extremal problems for the central projection

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Abstract

We consider a projection from the center of the unit sphere to a tangent space of it, the *central projection*, and study two area minimizing problems of the image of a closed subset in the sphere. One of the problems is the uniqueness of the tangent plane that minimizes the area for an arbitrary fixed subset. The other is the shape of the subset that minimizes the minimum value of the area. We also study the similar problems for the hyperbolic space.

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1 Introduction

Let S^n denote the n-dimentional unit sphere, $S^n = \left\{ x \in \mathbb{R}^{n+1} \middle| \, |x| = 1 \right\}$, σ_n the spherical standard measure on S^n and $\cdot$ the standard inner product of $\mathbb{R}^{n+1}$. Let Ω be an n-dimentional closed subset of S^n. The *polar set* of Ω is defined by

$$\Omega^* = \bigcap_{y \in \Omega} \{ x \in S^n | \, x \cdot y \leq 0 \} .$$

In the following, we always assume that Ω^* is not empty, i.e. Ω is contained in a hemisphere. Let p_x ($x \in -\mathring{\Omega}^*$, where $\mathring{\Omega}^*$ denotes the interior of the polar set Ω^*) be the projection from Ω to the tangent space of S^n at x and A_Ω the map that assigns the area of $p_x(\Omega)$ to a point x in $-\mathring{\Omega}^*$:

$$p_x : \Omega \ni y \mapsto \frac{y}{x \cdot y} \in T_x S^n, \quad A_\Omega(x) = \mathrm{Area}(p_x(\Omega)).$$

This projection p_x is used for making a (local) world atlas and also called the *gnomonic projection* when $n = 2$ in geography. Since the Jacobian of $p_x(y)$ is given by $Jp_x(y) = \dfrac{1}{(x \cdot y)^{n+1}}$, we have

$$A_\Omega(x) = \int_\Omega \frac{1}{(x \cdot y)^{n+1}} d\sigma_n(y).$$

F. Gao, D. Hug and R. Schneider showed the existence of a point that attains the minimum value of A_Ω and gave the characterization of it, but did not discuss the uniqueness in [1]. The existence of such a point follows from the fact that A_Ω is continuous on $-\mathring{\Omega}^*$ and diverges to $+\infty$ as x approaches the boundary of $-\Omega^*$. Direct calculation shows that if x attains the minimum value of A_Ω, then x satisfies the following formula:

$$\int_\Omega \frac{y}{(x \cdot y)^{n+2}} d\sigma_n(y) = A_\Omega(x) x.$$

In this paper we show the uniqueness of the A_Ω minimizer and estimate the minimum value of A_Ω for a

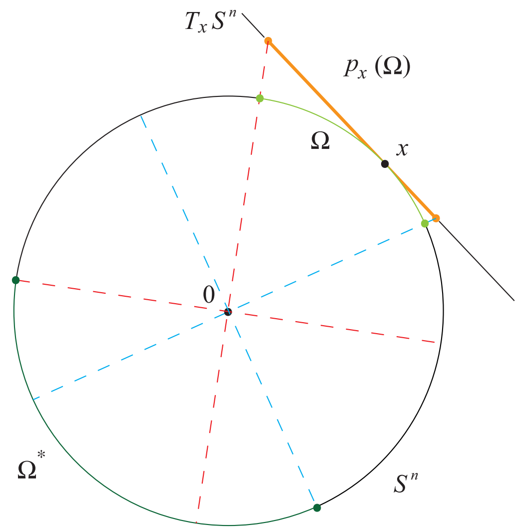

Figure 1: The polar set of Ω and $p_x(\Omega)$.

closed subset Ω in S^n having the same area with a disc in S^n. We also study the similar problems for the hyperbolic space.

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2 Extremal value of $A_\Omega(x)$

Let Ω and A_Ω be as in the introduction.

Theorem 2.1 *A_Ω has a unique minimum point.*

Proof. By using the polar coordinate, we can put

$$\begin{aligned} x &= x(\theta_1, \dots, \theta_n) \\ &= (\cos\theta_1, \sin\theta_1\cos\theta_2, \dots, \sin\theta_1\cdots\sin\theta_{n-1}\cos\theta_n, \sin\theta_1\cdots\sin\theta_{n-1}\sin\theta_n), \end{aligned}$$

where $(\theta_1, \dots, \theta_{n-1}, \theta_n) \in [0,\pi] \times \cdots \times [0,\pi] \times [0, 2\pi]$. Since $\dfrac{\partial^2 x}{\partial \theta_1^2} = -x$, we have

$$\frac{\partial^2 A_\Omega}{\partial \theta_1^2}(x) = (n+1)\int_\Omega \frac{(x\cdot y)^2 + (n+2)(\frac{\partial x}{\partial \theta_1}\cdot y)^2}{(x\cdot y)^{n+3}} d\sigma_n(y) > 0$$

on $-\mathring{\Omega}^*$. Suppose that there exist two points x' and x'' that attain the minimum value of A_Ω. By a rotation of S^n, we may assume that the sub-arc between x' and x'' of the great circle constant θ_i coordinates for $i \neq 1$. Then we have

$$\frac{\partial A_\Omega}{\partial \theta_1}(x') = \frac{\partial A_\Omega}{\partial \theta_1}(x'') = 0,$$

which is a contradiction. $\square$

Corollary 2.2 *If Ω is point symmetric at x_s, then x_s is the unique minimum point of A_Ω.*

By Corollary 2.2, the center of a disc D in S^n with non-empty polar set is the unique minimum point of A_D. This fact can also be indicated by the *moving plane method* (cf. [2]) became of the symmetry of the integrand of the partial derivative of A_Ω.

Theorem 2.3 *Let D be a disc in S^n with non-empty polar set. If $\sigma_n(\Omega) = \sigma_n(D)$, then*

$$\min_{x \in -\overset{\circ}{D}{}^*} A_D(x) \leq \min_{x \in -\overset{\circ}{\Omega}{}^*} A_\Omega(x)$$

and that equality holds if and only if Ω is a spherical cap.

Proof. By a translation of S^n, we may assume that the center of D coincides with the (unique) minimum point of A_Ω. Let x_c denote the center of D. Then we obtain

$$A_\Omega(x_c) - A_D(x_c) = \int_{\Omega \backslash (\Omega \cap D)} \frac{1}{(x_c \cdot y)^{n+1}} d\sigma_n(y) - \int_{D \backslash (\Omega \cap D)} \frac{1}{(x_c \cdot y)^{n+1}} d\sigma_n(y) \geq 0$$

since the following inequality holds for $y' \in \Omega \backslash (\Omega \cap D)$ and $y'' \in D \backslash (\Omega \cap D)$:

$$x_c \cdot y' = \cos \angle(x_c, y') \leq \cos \angle(x_c, y'') = x_c \cdot y'.$$

That equality holds if and only if $\sigma_n(\Omega \backslash (\Omega \cap D)) = 0$, namely, Ω is a disc in S^n. $\square$

We can see the following theorem with the same argument of Theorem 2.3.

Theorem 2.4 *Let D be a disc in S^n and $f : [0, +\infty) \to \mathbb{R}$ a strictly increasing continuous function. For an n-dimentional closed subset K in S^n, if $\sigma_n(K) = \sigma_n(D)$, then we have*

$$\min_{x \in S^n} \int_D f(\mathrm{dist}_{S^n}(x, y)) d\sigma_n(y) \leq \min_{x \in S^n} \int_K f(\mathrm{dist}_{S^n}(x, y)) d\sigma_n(y).$$

3 Hyperbolic case

Let $\langle \cdot, \cdot \rangle$ denote the indefinite inner product of $\mathbb{R}^{n+1}$ given by $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$, $\mathbb{R}^{n+1}_1$ the $(n+1)$-dimensional Euclidean space with $\langle \cdot, \cdot \rangle$, and $\mathbb{H}^n$ the Lorents model of the n-dimensional hyperbolic space, $\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1}_1 \,\middle|\, \langle x, x \rangle = -1, x_{n+1} > 0 \right\}$. Let Ω be an n-dimensional compact subset in $\mathbb{H}^n$ with non-empty interior. We define the maps p_x ($x \in \mathbb{H}^n$) and A_Ω by the similar way with the spherical case:

$$p_x : \Omega \ni y \mapsto -\frac{y}{\langle x, y \rangle} \in T_x \mathbb{H}^n, \quad A_\Omega(x) = \mathrm{Area}(p_x(\Omega)).$$

Since the Jacobian of $p_x(y)$ is given by $Jp_x(y) = \dfrac{1}{(-\langle x, y \rangle)^{n+1}}$, we have

$$A_\Omega(x) = \int_\Omega \frac{1}{(-\langle x, y \rangle)^{n+1}} d\mu_n(y),$$

where μ_n is the standard hyperbolic measure on $\mathbb{H}^n$.
The existence of a A_Ω maximizer follows from the fact that A_Ω is continuous on $\mathbb{H}^n$ and converges to 0 as x_{n+1} goes to $+\infty$. The similarly computation with [1] shows that if x attains the maximum value of A_Ω, then x satisfies the following formula:

$$\int_\Omega \frac{y}{(-\langle x, y \rangle)^{n+2}} d\mu_n(y) = A_\Omega(x) x.$$

The uniqueness of such a point does not always hold. For example, the following set Ω in $\mathbb{H}^1$ has two maximum points of A_Ω:

$$\Omega = \{\, (\sinh\theta, \cosh\theta) | -2 \leq \theta \leq -1 \} \cup \{\, (\sinh\theta, \cosh\theta) | \, 1 \leq \theta \leq 2 \} \, .$$

On the other hand, we can obtain the following theorems corresponding to Theorem 2.3 and 2.4 with the same arguments.

Theorem 3.1 *Let D be a disc in $\mathbb{H}^n$. If $\mu_n(\Omega) = \mu_n(D)$, then we have*

$$\max_{x\in\mathbb{H}^n} A_\Omega(x) \leq \max_{x\in\mathbb{H}^n} A_D(x)$$

and that equality holds if and only if Ω is a disc in $\mathbb{H}^n$.

Theorem 3.2 *Let D be a disc in $\mathbb{H}^n$ and $f : [0, +\infty) \to \mathbb{R}$ a strictly decreasing continuous function. For an n-dimentional compact subset K in $\mathbb{H}^n$, if $\mu_n(K) = \mu_n(D)$, then we have*

$$\max_{x\in\mathbb{H}^n} \int_K f(\mathrm{dist}_{\mathbb{H}^n}(x,y)) d\mu_n(y) \leq \max_{x\in\mathbb{H}^n} \int_D f(\mathrm{dist}_{\mathbb{H}^n}(x,y)) d\mu_n(y).$$

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